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Extending the multivariate generalised t and generalised VG distributions

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ABSTRACT

The *GGH* family of multivariate distributions is obtained by scale mixing on the Exponential Power distribution using the Extended Generalised Inverse Gaussian distribution. The resulting *GGH* family encompasses the multivariate generalised hyperbolic (*GH*), which itself contains the multivariate t and multivariate Variance-Gamma (*VG*) distributions as special cases. It also contains the generalised multivariate t distribution [O. Arslan, Family of multivariate generalised t distribution, Journal of Multivariate Analysis 89 (2004) 329–337] and a new generalisation of the *VG* as special cases. Our approach unifies into a single *GH*-type family the hitherto separately treated t-type [O. Arslan, A new class of multivariate distribution: Scale mixture of Kotz-type distributions, Statistics and Probability Letters 75 (2005) 18–28; O. Arslan, Variance–mean mixture of Kotz-type distributions, Communications in Statistics-Theory and Methods 38 (2009) 272–284] and *VG*-type cases. The *GGH* distribution is dual to the distribution obtained by analogous mixing on the scale parameter of a spherically symmetric stable distribution. Duality between the multivariate t and multivariate *VG* [S.W. Harrar, E. Seneta, A.K. Gupta, Duality between matrix variate t and matrix variate V.G. distributions, Journal of Multivariate Analysis 97 (2006) 1467–1475] does however extend in one sense to their generalisations.

1. Generalised Hyperbolic (GH_n) distribution

One of our foci in this paper is an approach which unifies into a single *GH*-type family the hitherto separately treated t-type [1,2] and *GH*-type [3] and the new *VG*-type cases. Investigation of duality between t-type and *VG*-type distributions, within the context of a unified *GH*-type family, in continuation of Harrar et al. [4] is the other focus.

Our starting point is the introduction of a convenient notation which makes clear that the pdf's of the multivariate t distribution and the multivariate Variance-Gamma (VG) distribution [5] of a random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$, with $n \geq 1$ may be regarded as special cases, rather than limit cases, of the multivariate Generalised Hyperbolic (*GH*) distribution (denoted by GH_n) of Barndorff-Nielsen [6].

For $\tau \in \mathbb{R}$, $a, b \geq 0$, with a and b not simultaneously 0, define $\bar{K}_\tau(a, b)$ as

$$\bar{K}_\tau(a, b) = \begin{cases} \left(\dfrac{a}{b}\right)^\tau K_\tau(ab), & \text{for } \tau \in \mathbb{R}, a, b > 0; \\ b^{-2\tau}\Gamma(\tau)2^{\tau-1}, & \text{for } \tau, b > 0, \text{ when } a = 0; \\ a^{2\tau}\Gamma(-\tau)2^{-\tau-1}, & \text{for } a > 0 \text{ and } \tau < 0, \text{ when } b = 0, \end{cases} \tag{1}$$

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where $\Gamma(\cdot)$ is the gamma function. Here $K_\tau(\omega)$ is the modified Bessel function of the third kind [7] with index $\tau \in \mathbb{R}$, given for $\omega > 0$ by

$$K_\tau(\omega) = \frac{1}{2}\int_0^\infty \exp\left\{-\frac{\omega}{2}(u^{-1}+u)\right\} u^{\tau-1}\mathrm{d}u.$$

That the second and third components of the definition (1) are appropriate follows by continuity, since it is readily shown that

$$\lim_{w\to 0} \omega^\eta K_\eta(a\omega) = \Gamma(\eta)2^{\eta-1}a^{-\eta}, \quad \text{for } a, \eta > 0.$$

We emphasize that, through the introduction of the $\bar{K}_\tau(a, b)$ notation, we especially intend to encompass and unify situations where one of a, b is 0, which corresponds to the *VG*-type and t-type special cases. The very recent results of Arslan [3], for example, partially overlap with ours, but he considers $a > 0$, $b > 0$, and obtains the t case by letting $b \to 0$.

We shall also need the following properties of $\bar{K}_\tau(a, b)$ which follow by using some results of $K_\tau(\cdot)$ from McLachlan [8] and Hurst et al. [9]

$$\begin{aligned}\bar{K}_{-\tau}(a, b) &= \bar{K}_\tau(b, a), \quad \text{for } a, b > 0\\ \frac{\mathrm{d}}{\mathrm{d}a}\bar{K}_\tau(a, b) &= -a\bar{K}_{\tau-1}(a, b), \quad \tau > 0\\ \frac{\mathrm{d}}{\mathrm{d}b}\bar{K}_\tau(a, b) &= -b\bar{K}_{\tau+1}(a, b); \quad \tau > 0.\end{aligned} \tag{2}$$

The pdf of the $GH_n(\mu, \Sigma, a, b, p)$ distribution is expressed as

$$f_{GH_n}(\mathbf{x}) = \frac{|\Sigma|^{-\frac{1}{2}}\bar{K}_{p-\frac{n}{2}}((a^2+s)^{\frac{1}{2}}, b)}{(2\pi)^{\frac{n}{2}}\bar{K}_p(a, b)},$$

where $s = (\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)$, $\mathbf{x} \in \mathbb{R}^n$.

When $b = 0$, $p = -\frac{q}{2} < 0$ and $a = \sqrt{q}$, together with (1), the density reduces to the density of the multivariate t distribution:

$$\frac{\Gamma\left(\frac{q+n}{2}\right)}{\pi^{\frac{n}{2}}\Gamma\left(\frac{q}{2}\right)q^{\frac{n}{2}}}|\Sigma|^{-\frac{1}{2}}\left(1+\frac{(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)}{q}\right)^{-\left(\frac{q}{2}+\frac{n}{2}\right)}, \quad \mathbf{x} \in \mathbb{R}^n.$$

When $a = 0$, $p = \frac{1}{\nu} > 0$ and $b = \sqrt{\frac{2}{\nu}}$, together with (1), the density becomes

$$\frac{2}{(2\pi)^{\frac{n}{2}}\Gamma\left(\frac{1}{\nu}\right)\nu^{\frac{1}{\nu}}}|\Sigma|^{-\frac{1}{2}}\left(\frac{s}{2/\nu}\right)^{\frac{1}{2}\left(\frac{1}{\nu}-\frac{n}{2}\right)} K_{\frac{1}{\nu}-\frac{n}{2}}\left(\left(\frac{2}{\nu}s\right)^{\frac{1}{2}}\right), \tag{3}$$

the density of multivariate *VG* distribution.

A random variable Y is said to have a (univariate) Generalised Inverse Gaussian (*GIG*) distribution, denoted by $GIG(a, b, p)$, if it has density

$$\begin{aligned}f_{GIG}(y) &= \frac{1}{2\bar{K}_p(a, b)}y^{p-1}\exp\left(-\frac{1}{2}(a^2y^{-1}+b^2y)\right), \quad y > 0;\\ &= 0, \quad \text{otherwise};\end{aligned} \tag{4}$$

where

$$\begin{cases} p \in \mathbb{R}, & \text{if } a, b > 0;\\ p, b > 0, & \text{if } a = 0;\\ p < 0, a > 0 & \text{if } b = 0.\end{cases}$$

It was demonstrated in Barndorff-Nielsen [6], that the multivariate Generalised Hyperbolic (GH_n) distribution is obtained by "variance mixing" of the multivariate normal with a *GIG* distribution, i.e. is the distribution of $\mathbf{X} = \mu + \Sigma^{\frac{1}{2}}Y^{\frac{1}{2}}\mathbf{Z}$, where $\mathbf{Z} \sim N_n(0, I_n)$ independently distributed of $Y \sim GIG(a, b, p)$.

Now, Arslan [1] introduced the multivariate generalised t distribution (GT_n) as a scale mixture of the multivariate exponential power and inverse generalised gamma distribution and with pdf:

$$\frac{\beta\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}B\left(\frac{q}{2}, \frac{n}{2\beta}\right)q^{\frac{n}{2\beta}}}|\Sigma|^{-\frac{1}{2}}\left(1+\frac{s^\beta}{q}\right)^{-\left(\frac{q}{2}+\frac{n}{2\beta}\right)}, \tag{5}$$

where $s = (\mathbf{x} - \mu)^T \Sigma^{-1}(\mathbf{x} - \mu)$ and $\mathbf{x} \in \mathbb{R}^n$. This has the multivariate t distribution as a special case when $\beta = 1$. The corresponding univariate version was first introduced by McDonald and Newey [10]. Consequently, one should be able to extend the GH_n distribution to encompass generalised multivariate t distribution as a special case, and also a (new) generalised multivariate VG distribution as a special case.

The univariate and multivariate VG distributions were first discussed as financial models in [5]. The univariate (symmetric) VG distribution as a model for log price increments of a single stock was introduced by these authors as a direct competitor to the Praetz t model in regard to kurtosis structure and analytical simplicity. While the GH_n distribution is more flexible, its two special cases remain popular. The univariate generalised t has already found related application [11, 12]. The multivariate VG distribution has also found application in financial context [13]. It is important to develop a unified context encompassing the generalisation of both in a way analogous to the mixing procedures giving rise to GH_n, to take applications further.

Further, GH_n possesses a "dual" property studied in [4]. If a spherically symmetric distribution has a "dual", then this property indicates its probability density function (pdf) is the characteristic function of a probability distribution family, apart from multiplicative constants. The duality property is unaffected by change of scale. A parametrised family of densities may be dual to itself in the sense that for any member of the family having a dual, that dual is also a member of the family. In this sense the GH_n family is dual to itself. A consequence of this is that a (spherically symmetric) multivariate t distribution satisfying certain conditions and a multivariate VG distribution are dual to each other. This is a striking property of these special cases whose significance will be further explored in the present paper. We investigate to what extent such dualities hold for our generalisation of the multivariate GH family. In this sense the present paper is also a sequel to Harrar et al. [4], although their theme was developed in a more general matrix-variate setting, with $\mathbf{X}$ being an $n \times p$ random matrix, and Y correspondingly $p \times p$, whereas we continue in this paper to confine ourselves, for simplicity, to the case with $p = 1$.

2. Auxiliary distributions

The required mixing distribution is a power transform of the Generalised Inverse Gaussian (GIG) distribution, defined by (4), which is first considered by Jørgensen [14] and more recently studied by Silva et al. [15], and is called the Extended Generalised Inverse Gaussian ($EGIG$) distribution when $a > 0$ and $b > 0$ in (6). We shall nevertheless use the same terminology even when one of a, b is 0. Let X be a random variable such that $X = Y^{\frac{1}{\lambda}}$, where $Y \sim GIG(a, b, p)$ and $\lambda > 0$. Then X has density

$$\begin{aligned} f_{EGIG}(x) &= \frac{\lambda}{2\bar{K}_p(a,b)} x^{\lambda p-1} \exp\left(-\frac{1}{2}(a^2x^{-\lambda} + b^2x^{\lambda})\right), \quad x > 0; \\ &= 0, \quad \text{otherwise}; \end{aligned} \tag{6}$$

where $\lambda > 0$; and $p \in \mathbb{R}$, if $a, b > 0$; $p, b > 0$, if $a = 0$; $p < 0, a > 0$ if $b = 0$. It is denoted by $EGIG(\lambda, a, b, p)$.

Notice that if $Y \sim EGIG(\lambda, a, b, p)$ then $1/Y \sim$EGIG$(\lambda, b, a, -p)$. In this sense the $EGIG$ distribution is dual to itself.

$EGIG$ has the Generalised Gamma distribution, originally introduced by Stacy [16], as a special case. A random variable Y is said to have a Generalised Gamma (GG) distribution, denoted by $GG(\lambda, \beta, p)$, if it has density

$$\begin{aligned} f_{GG}(y) &= \frac{|\lambda|\beta^{|\lambda|p}}{\Gamma(p)} y^{\lambda p-1} e^{-\beta^{|\lambda|}y^{\lambda}}, \quad y > 0; \\ &= 0, \quad \text{otherwise}; \end{aligned}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$, and $\beta, p > 0$. Using the definition of $\bar{K}(\cdot,\cdot)$ as in (1) we see that $EGIG(\lambda, 0, b, p) \stackrel{d}{=} GG(\lambda, (b^2/2)^{\frac{1}{\lambda}}, p)$, for $\lambda, b, p > 0$ and $EGIG(\lambda, a, 0, p) \stackrel{d}{=} GG(-\lambda, (a^2/2)^{\frac{1}{|\lambda|}}, -p)$, for $\lambda, a > 0, p < 0$.

Another distribution which is needed later in Section 5 is defined in terms of the product of a GG and an independent $EGIG$ random variable. Let Y be a positive random variable such that

$$Y \stackrel{d}{=} UV,$$

where $U \sim GG(\lambda, 1, p)$, $\lambda \in \mathbb{R} \setminus \{0\}$, $p > 0$ distributed independently of $V \sim EGIG(|\lambda|, a, b, q)$, for $q \in \mathbb{R}$, if $a, b > 0$; $q, b > 0$, if $a = 0$; $q < 0, a > 0$ if $b = 0$. Then Y has density

$$\begin{aligned} f_{GP}(y) &= \begin{cases} \dfrac{\lambda}{\Gamma(p)\bar{K}_q(a,b)} y^{\lambda p-1}\bar{K}_{q-p}((a^2+2y^{\lambda})^{\frac{1}{2}}, b), & \lambda > 0; \\ \dfrac{|\lambda|}{\Gamma(p)\bar{K}_q(a,b)} y^{\lambda p-1}\bar{K}_{q+p}(a, (b^2+2y^{\lambda})^{\frac{1}{2}}), & \lambda < 0; \end{cases} \quad y > 0; \\ &= 0, \quad \text{otherwise}. \end{aligned} \tag{7}$$

We say that a random variable with density as in (7) has a Generalised Product (GP) distribution, denoted more explicitly by $GP(\lambda, a, b, p, q)$. A GP distribution generalises the distribution of the product and of the ratio of two independent GG

random variables, which were both studied by Steece [17]. They can be obtained respectively by setting $a = 0$ for $\lambda > 0$ or $b = 0$ for $\lambda < 0$; and $b = 0$ for $\lambda > 0$ or $a = 0$ for $\lambda < 0$, as follows from the definition of $\bar{K}(\cdot, \cdot)$ in (1).

3. "Generalised" Generalized Hyperbolic (GGH_n)

To obtain the generalisation of the GH_n distribution, the multivariate Kotz-type distribution is, for our purposes, a natural starting point. A random vector $\mathbf{Z} = (Z_1, \ldots, Z_n)^T$, with $n \geq 1$, has a n-dimensional Kotz-type distribution (see for example [18], pp. 76–81), with $\mathbf{0}$ mean, identity scale matrix, parameters λ, and β where $2\lambda + n > 2$ and $\beta \in (0, \infty)$, denoted by $Kt_n(0, I_n, \beta, \lambda)$, if it has probability density:

$$f_{Kt_n}(\mathbf{z}) = k[\mathbf{z}^T\mathbf{z}]^{\lambda-1}\exp(-[\mathbf{z}^T\mathbf{z}]^{\beta}), \quad \mathbf{z} \in \mathbb{R}^n \tag{8}$$

where $k = \frac{\beta\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}\Gamma(\frac{2\lambda+n-2}{2\beta})}$. When $\mathbf{X} = \mu + \Sigma^{\frac{1}{2}}\mathbf{Z}$, where $\mu \in \mathbb{R}^n$, and Σ is an $(n \times n)$ positive definite symmetric matrix, we say that $\mathbf{X} \sim Kt_n(\mu, \Sigma, \beta, \lambda)$.

Note that the n-dimensional Exponential Power (EP_n) distribution, introduced by Gómez, Gómez-Villegas and Marín [19], is the case $\lambda = 1$ of the Kotz-type distribution, i.e. $Kt_n(\mu, \Sigma, \beta, 1)$. The multivariate normal $N_n(\mu, 2\Sigma)$ is also a special case by further setting $\beta = 1$ from the EP_n, i.e. $Kt_n(\mu, \Sigma, 1, 1)$.

A generalisation of GH_n is obtained by mixing a Kt_n distribution with an $EGIG$ distribution. Notice that both distributions in this mixing generalise those in obtaining the pdf of the GH_n case.

Theorem 1. *Let $\mathbf{Z} \sim Kt_n(0, I_n, \beta, \lambda)$, $V \sim EGIG(\beta, a, b, p)$ be independently distributed random vector and scalar. Let $\mathbf{X}$ be a random vector which is defined by*

$$\mathbf{X} = \mu + \Sigma^{\frac{1}{2}}V^{\frac{1}{2}}\mathbf{Z},$$

where $\mu \in \mathbb{R}^n$, Σ is an $(n \times n)$ positive definite symmetric matrix. Then $\mathbf{X}$ is an elliptically distributed random vector with density:

$$f_X(\mathbf{x}) = \frac{\beta\Gamma(\frac{n}{2})s^{\lambda-1}\bar{K}_{p-\frac{\lambda-1}{\beta}-\frac{n}{2\beta}}((a^2+2s^{\beta})^{\frac{1}{2}}, b)}{\pi^{\frac{n}{2}}\Gamma\left(\frac{2\lambda+n-2}{2\beta}\right)|\Sigma|^{\frac{1}{2}}\bar{K}_p(a, b)}, \quad \begin{cases} \text{for } \beta > 0, p \in \mathbb{R}, \text{ if } a, b > 0; \\ \text{for } \beta, p, b > 0, \text{ if } a = 0; \\ \text{for } p < 0, a, \beta > 0 \text{ if } b = 0; \end{cases} \tag{9}$$

where $\lambda > \frac{2-n}{2}$ and $s = (\mathbf{x} - \mu)^T\Sigma^{-1}(\mathbf{x} - \mu)$ and $\mathbf{x} \in \mathbb{R}^n$.

Proof. As $\mathbf{Z}$ and V are independently distributed, we have $\mathbf{X}|V \sim Kt_n(\mu, V\Sigma, \beta, \lambda)$. Using (6) the density function of the random vector $\mathbf{X}$ is

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\beta\Gamma\left(\frac{n}{2}\right)s^{\lambda-1}\bar{K}_{p-\frac{\lambda-1}{\beta}-\frac{n}{2\beta}}((a^2+2s^{\beta})^{\frac{1}{2}}, b)}{\pi^{\frac{n}{2}}\Gamma\left(\frac{2\lambda+n-2}{2\beta}\right)|\Sigma|^{\frac{1}{2}}\bar{K}_p(a, b)}.$$

Now, there is a non-negative function g such that $f_{\mathbf{X}}(\mathbf{x}) = |\Sigma|^{-\frac{1}{2}}g((\mathbf{x} - \mu)^T\Sigma^{-1}(\mathbf{x} - \mu))$, i.e.

$$g(t^2) = \frac{\beta\Gamma\left(\frac{n}{2}\right)t^{2(\lambda-1)}\bar{K}_{p-\frac{\lambda-1}{\beta}-\frac{n}{2\beta}}((a^2+2t^{2\beta})^{\frac{1}{2}}, b)}{\pi^{\frac{n}{2}}\Gamma\left(\frac{2\lambda+n-2}{2\beta}\right)\bar{K}_p(a, b)}, \quad t > 0;$$

which satisfies

$$\int_0^{\infty} t^{\frac{n}{2}-1}g(t)\mathrm{d}t < \infty, \quad \begin{cases} \text{for } \beta > 0, p \in \mathbb{R}, \text{ if } a, b > 0; \\ \text{for } \beta, p, b > 0, \text{ if } a = 0; \\ \text{for } p < 0, a, \beta > 0 \text{ if } b = 0, \end{cases}$$

while $\lambda > \frac{2-n}{2}$ by comparing with the form of (7). Hence $g(\cdot)$ is the density generator for the elliptical distributed random vector $\mathbf{X}$ and, in terms of the notation for such functions, is denoted by $X \sim E_n(\mu, \Sigma, g)$ [20, p. 70]. □

For $a > 0$ and $b > 0$ and $\lambda > \frac{2-n}{2}$ the distribution of X is derived and labelled the Symmetric Extended Generalised Hyperbolic (Definition 3.4 of Arslan [3].) The case $b = 0$ is treated individually as the scale mixture of the Kotz-type distribution in [2], and is thus a generalisation even of the GT_n distribution. The GT_n, with pdf (5) as in [1] is the case $\lambda = 1$.

Theorem 1 incorporates another scale mixture of the Kotz-type distribution. When $a = 0$, (9) becomes

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\beta\Gamma\left(\frac{n}{2}\right) s^{\lambda-1}\bar{K}_{p-\frac{\lambda-1}{\beta}-\frac{n}{2\beta}}\left(\sqrt{2}s^{\frac{\beta}{2}}, b\right)}{\pi^{\frac{n}{2}}\Gamma\left(\frac{2\lambda+n-2}{2\beta}\right)|\Sigma|^{\frac{1}{2}} b^{-2p}\Gamma(p)2^{p-1}}$$

which is thus a *VG* analogue to the distribution considered in [2].

We shall, however be focussing in general on the case $\lambda = 1$, which is still to be more fully explored in a unified setting (that is, when one or other of a, b may be zero).

Thus we now define the generalisation of the generalised hyperbolic *GH* distribution given by (9) with $\lambda = 1$.

Definition 1. A random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$, with $n \geq 1$, is said to have an n-dimensional generalised GH_n distribution, denoted by $GGH_n(\mu, \Sigma, \beta, a, b, p)$, if its density is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{n\Gamma\left(\frac{n}{2}\right)\bar{K}_{p-\frac{n}{2\beta}}((a^2+2s^{\beta})^{\frac{1}{2}}, b)}{2\pi^{\frac{n}{2}}\Gamma\left(1+\frac{n}{2\beta}\right)|\Sigma|^{\frac{1}{2}}\bar{K}_p(a,b)}, \quad \begin{cases} \text{for } \beta > 0, p \in \mathbb{R}, \text{if } a, b > 0; \\ \text{for } \beta, p, b > 0, \text{if } a = 0; \\ \text{for } p < 0, a, \beta > 0 \text{ if } b = 0; \end{cases}$$

with $X \sim E_n(\mu, \Sigma, g)$ where

$$g(t^2) = \frac{n\Gamma\left(\frac{n}{2}\right)\bar{K}_{p-\frac{n}{2\beta}}((a^2+2t^{2\beta})^{\frac{1}{2}}, b)}{2\pi^{\frac{n}{2}}\Gamma\left(1+\frac{n}{2\beta}\right)\bar{K}_p(a,b)}, \quad t > 0. \tag{10}$$

Hence generalisation of GH_n to GGH_n is obtained by scale mixing an EP_n distribution by an *EGIG* distribution. Arslan [2] considered scale mixture of the Kotz-type distribution with an inverse *GG* distribution. Arslan [3] takes this mixing concept even further by considering the mean–variance mixture the Kotz-type distribution with the *EGIG* distribution with $a > 0$ and $b > 0$ (see his Definition 3.3), which is capable of incorporating skewness into the distribution. But we are largely focussed on the two special cases when one of a, b is zero. Another reason, discussed shortly, is duality, which we shall show does not extend beyond the case $\lambda = 1$.

In the $GGH_n(\mu, \Sigma, \beta, a, b, p)$ distribution, β and p are shape parameters; a and b are scale parameters. When $\beta = 1$, $a = \sqrt{2}a^*$ and $b = b^*/\sqrt{2}$, the $GH_n(\mu, \Sigma, a^*, b^*, p)$ distribution is obtained. When $b = 0$, $p = -\frac{q}{2} < 0$ and $a = \sqrt{2q}$, together with (1), the density reduces to (5), which is the density of the multivariate generalised t distribution, introduced by Arslan [1]. When $a = 0$, $p = \frac{1}{\nu} > 0$ and $b = \sqrt{\frac{1}{\nu}}$, together with (1), the density becomes

$$\frac{n\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}\Gamma\left(1+\frac{n}{2\beta}\right)2^{\frac{n}{2\beta}}\Gamma\left(\frac{1}{\nu}\right)\nu^{\frac{1}{\nu}}}|\Sigma|^{-\frac{1}{2}}\bar{K}_{\frac{1}{\nu}-\frac{n}{2\beta}}\left(s^{\frac{\beta}{2}}, \sqrt{\frac{2}{\nu}}\right), \tag{11}$$

where $s = (\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)$ and $\mathbf{x} \in \mathbb{R}^n$. The pdf (11) is a density which encompasses the multivariate *VG* distribution, defined by (3) as a special case by setting $\beta = 1$. Thus (11) defines a generalisation of the multivariate *VG* in the same spirit as the generalised multivariate t.

When $n = 2$, we can study the density plot of the generalised *GH* distribution. In Fig. 1, we present the graphs for $\beta = 0.5$, 1, and 10, all with $\mu = \mathbf{0}$, $\Sigma = \mathbf{I}_2$, $a = 1$ and $b = 1$. The left column of Fig. 1 provides a three-dimensional impression of the density and the right column provides the corresponding cross-section view. One interesting observation is the shape of the density changes, as β increases, from a cusped form, then to bell shape and then to the uniform form. A pictorial study of the more general variance–mean mixing of the Kotz-type distribution can be found in [3].

4. Duality

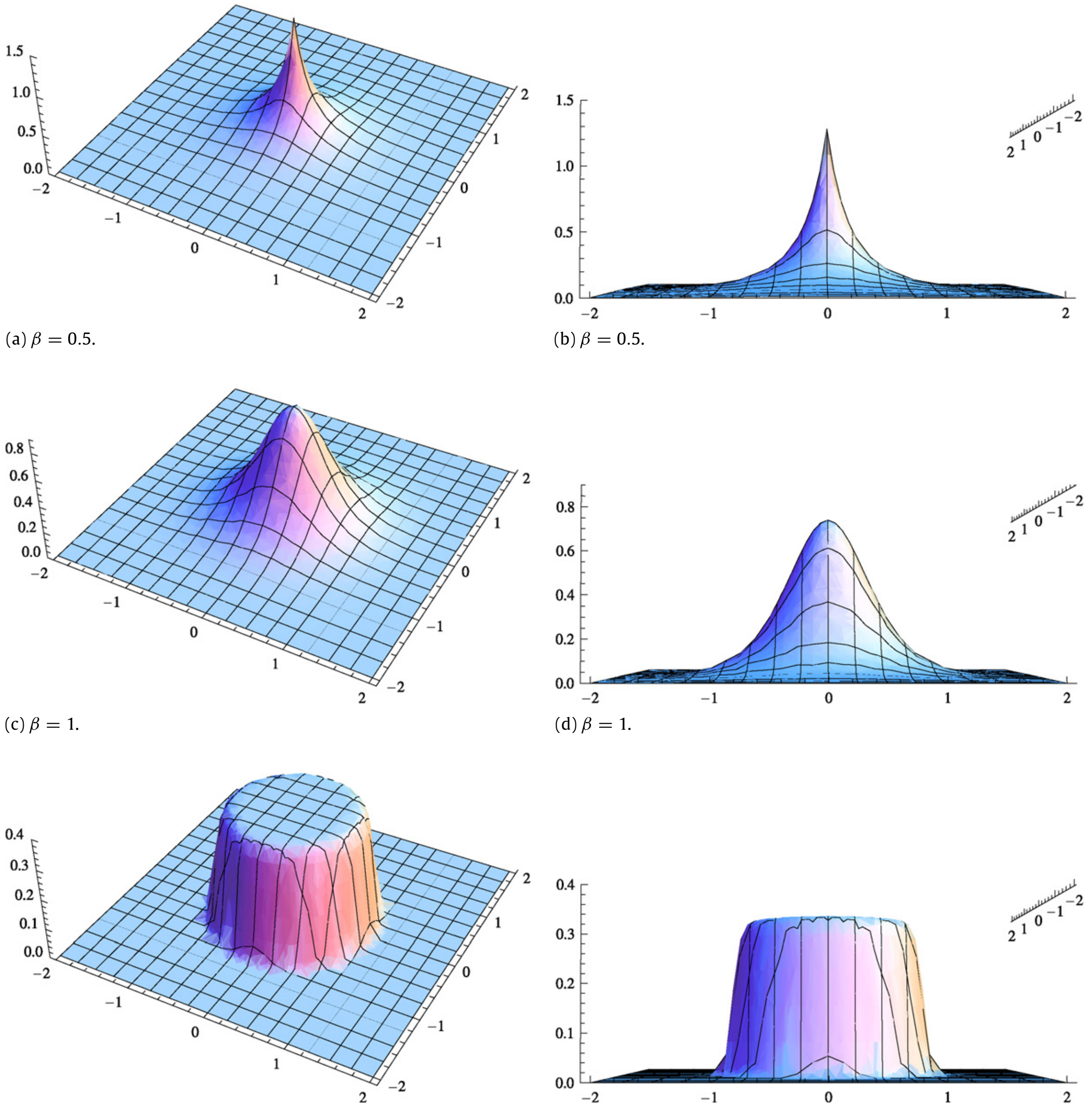
Although the theorem fundamental to the study of duality is given in [4] in a matrix setting, only a vector version is needed for our context.

Theorem 2 (*The Duality Theorem*). *Let $\mathbf{X} \in \mathbb{R}^n$ be random vector with characteristic function $\psi_1(\mathbf{t})$. If $\psi_1(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$ satisfies*

$$\psi_1(\mathbf{t}) \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^n} \psi_1(\mathbf{t})\mathrm{d}\mathbf{t} < \infty$$

(and so has bounded continuous pdf $f_1(\mathbf{x})$, symmetric about $\mathbf{0}$), then $f_1(\mathbf{0}) > 0$ and

$$f_2(\mathbf{w}) = \frac{\psi_1(\mathbf{w})}{\int_{\mathbb{R}^n}\psi_1(\mathbf{t})\mathrm{d}\mathbf{t}} \tag{12}$$



(a) $\beta = 0.5$. (b) $\beta = 0.5$.

(c) $\beta = 1$. (d) $\beta = 1$.

(e) $\beta = 10$. (f) $\beta = 10$.

Fig. 1. Pdf plot of $GGH_2(\mathbf{0}, \mathbf{I}_2, \beta, 1, 1, 3)$.

is a pdf over $\mathbb{R}^n$, whose characteristic function is

$$\psi_2(\mathbf{u}) = \frac{f_1(\mathbf{u})}{f_1(\mathbf{0})}, \quad \mathbf{u} \in \mathbb{R}^n.$$

The distribution of $\mathbf{X} \in \mathbb{R}^n$ and the distribution described by the pdf (12) are said to be dual to each other.

To proceed further, we give a generalisation of Theorem 1.2 of Harrar et al. [4], albeit again in random vector rather than random matrix context.

Theorem 3. *Let $\mathbf{X} \stackrel{d}{=} Y^{\frac{1}{2}}\mathbf{Z}$, where $Y > 0$ with pdf $f_Y(y)$, and $\mathbf{Z}$ is a random vector and they are independently distributed. Assume $\psi_{\mathbf{Z}}(\mathbf{t}) = E(\mathrm{e}^{\mathrm{i}\mathbf{t}^T\mathbf{Z}}) \geq 0$, $\int_{\mathbb{R}^n} \psi_{\mathbf{Z}}(\mathbf{t})\mathrm{d}\mathbf{t} < \infty$ and $EY^{-\frac{n}{2}} < \infty$. Then $\mathbf{X}$ has a dual r.v. $\mathbf{D}_{\mathbf{X}}$, whose pdf is*

$$f_{\mathbf{D}_{\mathbf{X}}}(\mathbf{t}) = \frac{E_Y\left(\psi_{\mathbf{Z}}\left(Y^{\frac{1}{2}}\mathbf{t}\right)\right)}{\left(E\left(Y^{-\frac{n}{2}}\right)\int_{\mathbb{R}^n}\psi_{\mathbf{Z}}(\mathbf{s})\mathrm{d}\mathbf{s}\right)}.$$

Moreover, $\mathbf{D_X} \stackrel{d}{=} W^{\frac{1}{2}}\mathbf{D_Z}$*, where* $\mathbf{D_Z}$ *is a r.v. with pdf describing the dual distribution of* $\mathbf{Z}$*,* $W > 0$ *and* $\mathbf{Z}$ *are independently distributed, and* W *has pdf*

$$f_W(w) = \frac{w^{\frac{n}{2}-2} f_Y(w^{-1})}{EY^{-\frac{n}{2}}}, \quad w > 0;$$
$$= 0 \quad \textit{else;}$$

so that $EW^{-\frac{n}{2}} = \frac{1}{EY^{-\frac{n}{2}}} < \infty$.

Proof. The characteristic function of $\mathbf{X}$, $\psi_\mathbf{X}$, satisfies

$$\psi_\mathbf{X}(\mathbf{t}) = E_Y\left(\psi_\mathbf{Z}\left(Y^{\frac{1}{2}}\mathbf{t}\right)\right) \geq 0, \quad \text{and} \quad \int_{\mathbb{R}^n} \psi_\mathbf{X}(\mathbf{t})\mathrm{d}\mathbf{t} = EY^{-\frac{n}{2}} \int_{\mathbb{R}^n} \psi_\mathbf{Z}(\mathbf{t})\mathrm{d}\mathbf{t} < \infty,$$

as both $EY^{-\frac{n}{2}}$ and $\int_{\mathbb{R}^n} \psi_\mathbf{Z}(\mathbf{t})\mathrm{d}\mathbf{t} < \infty$. Then, by Theorem 2, a r.v. $\mathbf{D_X}$ whose distribution is the dual of $\mathbf{X}$ exists and has pdf of the form

$$f_{\mathbf{D_X}}(\mathbf{t}) = \frac{E_Y\left(\psi_\mathbf{Z}\left(Y^{\frac{1}{2}}\mathbf{t}\right)\right)}{\left(E\left(Y^{-\frac{n}{2}}\right)\int_{\mathbb{R}^n}\psi_\mathbf{Z}(\mathbf{s})\mathrm{d}\mathbf{s}\right)} = \int_0^\infty \frac{1}{w^{\frac{n}{2}}} f_{\mathbf{D_Z}}\left(\frac{\mathbf{t}}{w^{\frac{1}{2}}}\right) f_W(w)\mathrm{d}w$$

where

$$f_{\mathbf{D_Z}}(\mathbf{t}) = \frac{\psi_\mathbf{Z}(\mathbf{t})}{\int_{\mathbb{R}^n}\psi_\mathbf{Z}(\mathbf{s})\mathrm{d}\mathbf{s}}$$

is the pdf of a r.v. $\mathbf{D_Z}$ whose distribution is the dual of $\mathbf{Z}$ by Theorem 2 and

$$f_W(w) = \frac{w^{\frac{n}{2}-2} f_Y\left(\frac{1}{w}\right)}{EY^{-\frac{n}{2}}} \geq 0, \quad w > 0 \quad \text{and} \quad \int_0^\infty \frac{w^{\frac{n}{2}-2} f_Y\left(\frac{1}{w}\right)}{EY^{-\frac{n}{2}}}\mathrm{d}w = \frac{\int_0^\infty y^{-\frac{n}{2}} f_Y(y)\mathrm{d}y}{EY^{-\frac{n}{2}}} = 1,$$

so f_W is a proper pdf say for a random variable W. As a result, $\mathbf{D_X} \stackrel{d}{=} W^{\frac{1}{2}}\mathbf{D_Z}$ where $\mathbf{D_Z}$ and W are independently distributed. It then follows directly that

$$EW^{-\frac{n}{2}} = \frac{1}{EY^{-\frac{n}{2}}} < \infty. \quad \square$$

The condition that $Y > 0$ is described by a positive density on $y > 0$ may be relaxed, and is made for simplicity.

Corollary 1. $Y \sim EGIG(\lambda, a, b, p)$ *with* $EY^{-\frac{n}{2}} < \infty$ *if and only if* $W \sim EGIG(\lambda, b, a, \frac{n}{2\lambda} - p)$ *with* $EW^{-\frac{n}{2}} < \infty$.

Theorem 2 permits one to begin with a characteristic function satisfying its conditions to provide the pdf of the dual distribution. To show that a parametrised family of distributions is dual to itself by this means, one must have available also the explicit form of $f_1(\mathbf{x})$, by means of which one may compare the characteristic function $\psi_2(\mathbf{t})$ with $\psi_1(\mathbf{t})$. Harrar et al. [4] demonstrate that GH_n family is dual to itself by this means, since its pdf is also its characteristic function apart from multiplicative and scale constants, and from change of parameter values.

In particular it follows from this result that both the multivariate t and VG distributions satisfying the conditions of Theorem 2 are dual to each other.

The characteristic function of $GGH_n(\mathbf{0}, I_n, \beta, a, b, p)$ is found in our Section 6, in rather complex form. Providing the conditions of Theorem 2 are satisfied, this will give the pdf of the dual distribution, although in far-from-closed form. The reason for the complexity of the pdf of the dual will become apparent shortly.

Actually, we shall proceed in a reverse direction, as suggested by the structure of Theorem 1. Recall that a random vector $\mathbf{Z}$ has a spherical symmetric stable law (see [21]) with index $0 < \alpha \leq 2$, and scale parameter $\gamma > 0$ if its characteristic function has the form

$$\psi_{stable}(\mathbf{t}) = \mathrm{e}^{-(\gamma\mathbf{t}^T\mathbf{t})^{\frac{\alpha}{2}}}, \quad \mathbf{t} \in \mathbb{R}^n.$$

It is obvious that such stable law satisfies Theorem 2 and it is dual to EP_n for $\beta \leq 1$, by comparison with (8) while setting $\lambda = 1$.

Moreover, if: (1) we regard the pdf of a GGH as being the pdf $f_2(x)$ of the *dual* distribution; (2) then consider Theorem 1 where the mixing distribution is $EGIG$, and: (3) notice that the pdf of an EP_n distribution is functionally equivalent to the cf of the symmetric stable distribution, it is: (4) clear from Theorem 3 and Corollary 1 that the GGH distribution is dual to the distribution resulting from mixing of the symmetric stable distribution on its γ parameter with an appropriate $EGIG$ distribution.

We summarise the idea into the following theorem.

Theorem 4. *Define a random vector* $\mathbf{X}$ *as* $\mathbf{X} \stackrel{d}{=} Y^{\frac{1}{2}}\mathbf{Z}$, *where* $Y \sim EGIG(\frac{\alpha}{2}, a, b, p)$ *and is independently distributed of* $\mathbf{Z}$ *which has the spherically symmetric stable law with index* $0 < \alpha \leq 2$ *with* $\gamma = 1$. *Then if*

$$\{a > 0,\ b > 0,\ p \in \mathbb{R}\}$$
$$\left\{a = 0,\ b > 0,\ p > \frac{n}{\alpha}\right\}$$
$$\{a > 0,\ b = 0,\ p < 0\};$$

for $0 < \alpha \leq 2$, *a r.v.* $\mathbf{D_X}$ *whose distribution is the dual of* $\mathbf{X}$ *exists and* $\mathbf{D_X} \sim GGH_n(0, I_n, \frac{\alpha}{2}, b, a, -p + \frac{n}{\alpha})$.

Proof. Using (6) with $\lambda = \frac{\alpha}{2}$, the random vector $\mathbf{X}$ has characteristic function:

$$\psi_{\mathbf{X}}(\mathbf{t}) = \frac{\bar{K}_p(a, (b^2 + 2(\mathbf{t}^T\mathbf{t})^{\frac{\alpha}{2}})^{\frac{1}{2}})}{\bar{K}_p(a, b)}, \quad \mathbf{t} \in \mathbb{R}^n,$$

for $p \in \mathbb{R}$, and $a, b > 0$; $p, b > 0$, when $a = 0$; $a > 0, p < 0$, when $b = 0$.

Next, $EY^{-\frac{n}{2}} < \infty$ requires $a > 0, b > 0, p \in \mathbb{R}$; $a = 0, b > 0, p - \frac{n}{\alpha} > 0$; $a > 0, b = 0, p - \frac{n}{\alpha} < 0$. So overall, we need the conditions on a, b, p stated in the theorem.

Then by Theorem 3 a random vector $\mathbf{D_X}$ whose distribution is the dual of $\mathbf{X}$ exists under these conditions and has pdf of the form

$$\begin{aligned} f_{\mathbf{D_X}}(\mathbf{w}) &= \frac{\psi_{\mathbf{X}}(\mathbf{w})}{\int_{\mathbb{R}^n} \psi_{\mathbf{X}}(\mathbf{s})\mathrm{d}\mathbf{s}} \\ &= \frac{n\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}\Gamma\left(1+\frac{n}{\alpha}\right)\bar{K}_{-p+\frac{n}{\alpha}}(a, b)} \bar{K}_{(-p+\frac{n}{\alpha})-\frac{n}{\alpha}}\left(\left(b^2 + 2(\mathbf{w}^T\mathbf{w})^{\frac{\alpha}{2}}\right)^{\frac{1}{2}}, a\right), \end{aligned}$$

$\mathbf{w} \in \mathbb{R}^n$, which is the density of $GGH_n(0, I_n, \frac{\alpha}{2}, b, a, -p + \frac{n}{\alpha})$. $\square$

We remark that the pdf of $\mathbf{D_X}$ is meaningful for any $\alpha > 0$, which suggests the possible existence of the dual distribution of $GGH_n(0, I_n, \frac{\alpha}{2}, b, a, -p + \frac{n}{\alpha})$ even for $\alpha > 2$. However, even though $EW^{-\frac{n}{2}} < \infty$ holds under the constraints on a, b, p given in the theorem even when $\alpha > 2$, this does not guarantee the existence of a dual distribution since in general the conditions of Theorem 2 are not satisfied. For instance, the cf for $GGH_3(0, I_3, \frac{\alpha}{2}, b, a, -p + \frac{n}{\alpha})$ with $\alpha = 3, p = b = a = 1$ is not always non-negative on $\mathbb{R}^n$. In fact in this instance, using the cf given in our Section 6, Theorem 7, (1), the cf for $\mathbf{X} \sim GGH_3(0, I_3, 1.5, 1, 1, 0)$, namely $\psi_{\mathbf{X}}(\mathbf{t}) = -0.0016$ when $\sqrt{(\mathbf{t}^T\mathbf{t})} = 6.8$.

In the scalar case, for example, $X \sim GGH_1(0, 1, 1.5, 1, 1, 0)$, $\psi_X(4.5) = -0.0046$.

In any case, we conclude that the GGH_n distribution family is not self-dual in the sense of Theorem 2. On the other hand, the generalised multivariate t, which we shall now denote by GT_n, and the generalised multivariate *VG* may be regarded as dual to each other in the sense that the distribution of the mixing random variable in the sense of Theorem 1 with $\lambda = 1$, $V \sim EGIG\ (\beta, a, 0, p)$ with $p < 0$ giving the GT_n, has $1/V \sim EGIG(\beta, 0, a, -p)$, which is of the form of mixing variable V required to give the generalised multivariate *VG* (GVG). We are using here the fact that if $Y \sim EGIG(\lambda, a, b, p)$, then $1/Y \sim EGIG(\lambda, b, a, -p)$ as discussed in Section 2.

We see from Theorem 4 that a GGH_n distribution has a dual in the sense of Theorem 2. The scale mixture by *EGIG* of the more general Kotz-type distribution (except for its special case $\lambda = 1$ which leads to the GGH_n) does not possess a dual in the sense of Theorem 2. For suppose $\mathbf{X} = Y^{\frac{1}{2}}\mathbf{Z}$, where $\mathbf{Z}$ is the n-dimensional Kotz-type distribution and $Y > 0$ has independently distributed of $\mathbf{Z}$, then $\mathbf{X}$ has density

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}) &= \int_0^{\infty} \frac{1}{y^{\frac{n}{2}}} f_{\mathbf{Z}}\left(\frac{\mathbf{x}}{y^{\frac{1}{2}}}\right) f_Y(y)\mathrm{d}y \\ &= k[\mathbf{x}^T\mathbf{x}]^{\lambda-1} \int_0^{\infty} \frac{1}{y^{\frac{n}{2}+\lambda-1}} \exp(-s\{[\mathbf{x}^T\mathbf{x}]/y\}^{\beta}) f_Y(y)\mathrm{d}y. \end{aligned}$$

If $\lambda < 1$ or $\lambda > 1$ then $f_{\mathbf{X}}(\mathbf{x})$ is, respectively, unbounded, or 0 at $\mathbf{x} = \mathbf{0}$. Thus the characteristic function in either case cannot satisfy the conditions of Theorem 2, which imply $0 < f_{\mathbf{X}}(\mathbf{0}) < \infty$.

As GGH_n encompasses as a special case the GT_n, in the next two sections we will discuss a similar set of properties to those studied by Arslan [1] for GT_n. Our Theorems 5, 7 and 8 below generalise Propositions 2, 3 and 4 of Arslan [1]. Our Theorem 6 is entirely new.

5. Representations

From the theory in [20,22], it can be shown that GGH_n has two different stochastic representations, one involving $\mathbf{u}^{(n)}$ and the other $\mathbf{V}^{(n)}$, the random vectors uniformly distributed *on the surface* and *inside* of a unit sphere in $\mathbb{R}^{(n)}$ respectively.

Theorem 5. *The random vector* $\mathbf{X} \sim GGH_n(\mu, \Sigma, \beta, a, b, p)$ *if and only if*

$$\mathbf{X} \stackrel{d}{=} \mu + \Sigma^{\frac{1}{2}} R\mathbf{u}^{(n)}$$

where $R \sim GP(2\beta, a, b, \frac{n}{2\beta}, p)$ *and is distributed independently of* $\mathbf{u}^{(n)}$, *a random vector distributed uniformly on the surface of a unit sphere in* $\mathbb{R}^{(n)}$. *Moreover, for* $\beta > 0$ *and positive integer k,*

$$E(R^k) = \frac{\Gamma\left(\frac{n+k}{2\beta}\right) \bar{K}_{p+\frac{k}{2\beta}}(a, b)}{\Gamma\left(\frac{n}{2\beta}\right) \bar{K}_p(a, b)}$$

if $p \in \mathbb{R}$, *and* $a, b > 0$; $p, b > 0$, *when* $a = 0$; $a > 0, p < -\frac{k}{2\beta}$, *when* $b = 0$.

Proof. $\mathbf{X} \sim GGH_n(\mu, \Sigma, \beta, a, b, p) \sim E_n(\mu, \Sigma, g)$, where $g(r^2)$ as defined by (10). The results then follow by Theorem 2.5.3 on p. 57 and Theorem 2.5.5 on p. 59 of Fang and Zhang [20]. □

One important corollary of the above stochastic representation is that $(\mathbf{X} - \mu)^T \Sigma^{-1} (\mathbf{X} - \mu) \stackrel{d}{=} R^2$.

Besides the above standard stochastic representation for elliptical distributions, the GGH_n possess another type of representation due to the fact that $\Sigma^{-\frac{1}{2}}(\mathbf{X} - \mu)$ is unimodal (star-unimodal to be precise), i.e. the density is non-increasing when one moves away from the mode. This property implies that the GGH_n is also representable as a scale mixture of the uniform (SMU). The SMU property is characterised in [22].

Theorem 6. *The random vector* $\mathbf{X} \sim GGH_n(\mu, \Sigma, \beta, a, b, p)$ *if and only if*

$$\mathbf{X} \stackrel{d}{=} \mu + \Sigma^{\frac{1}{2}} W\mathbf{V}^{(n)},$$

where $W \sim GP(2\beta, a, b, \frac{n}{2\beta} + 1, p)$ *and is distributed independently of* $\mathbf{V}^{(n)}$, *a random vector distributed uniformly inside a unit sphere in* $\mathbb{R}^{(n)}$.

Proof. As $\mathbf{X} \sim E_n(\mu, \Sigma, g)$ where $g(r)$ was defined in (10) is differentiable on $r > 0$, and by using (2), we have

$$-\frac{\mathrm{d}}{\mathrm{d}r} g(r^2) = \frac{n\beta\Gamma(\frac{n}{2}) r^{2\beta-1}}{\pi^{\frac{n}{2}} \Gamma\left(1 + \frac{n}{2\beta}\right) \bar{K}_p(a, b)} \bar{K}_{p-\frac{n}{2\beta}-1}\left((a^2 + 2r^{2\beta})^{\frac{1}{2}}, b\right) \geq 0,$$

for all $r > 0$. It follows from Theorem 2 in [22] that $\mathbf{X} \sim GGH_n(\mu, \Sigma, \beta, a, b, p)$ if and only if $\mathbf{X} \stackrel{d}{=} \mu + \Sigma^{\frac{1}{2}} W\mathbf{V}^{(n)}$ and W has density of the form

$$f_W(w) = -\frac{2\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)} w^n \frac{\mathrm{d}g(w^2)}{\mathrm{d}w} = \frac{2\beta w^{2\beta(\frac{n}{2\beta}+1)-1}}{\Gamma\left(1 + \frac{n}{2\beta}\right) \bar{K}_p(a, b)} \bar{K}_{p-\left(\frac{n}{2\beta}+1\right)}\left((a^2 + 2w^{2\beta})^{\frac{1}{2}}, b\right),$$

which is the density of $GP(2\beta, a, b, \frac{n}{2\beta} + 1, p)$, from (7). □

Thus the GP distribution enters into both representations of this section. The slight difference in its two parametrisations is instructive.

6. Some fundamental properties

GT_n is a special case of the GGH_n as demonstrated in Section 3, by substituting in $b = 0$, $p = -\frac{q}{2} < 0$ and $a = \sqrt{2q}$, together with the notation (1). Accordingly, the results of Proposition 7 and 8 of Arslan [1] generalise as follows. Their proofs are, likewise, omitted. The results analogous to those for GT for GVG can then be obtained simply by substituting in $a = 0$, $p = \frac{1}{\nu}$ and $b = \sqrt{\frac{1}{\nu}}$. Proposition 2.1 and 2.2 of Arslan [3] partially overlap with our Theorem 8, but their focus is mainly on $N \neq 1$ and $\beta \neq 1$ whereas ours is the case of $N = 1$ and $\beta \neq 1$, using the parameter notation of Arslan [3].

Theorem 7. *If* $\mathbf{X} \sim GGH_n(\mu, \Sigma, \beta, a, b, p)$, *then*

(1) *It has characteristic function as*

$$\psi_{\mathbf{X}}(\mathbf{t}) = \frac{2\beta e^{i\mathbf{t}^T\mu}}{\Gamma\left(\frac{n}{2\beta}\right) \bar{K}_p(a, b)} \int_0^\infty \Phi_n\left(r\sqrt{\mathbf{t}^T \Sigma \mathbf{t}}\right) r^{n-1} \bar{K}_{p-\frac{n}{2\beta}}((a^2 + 2r^{2\beta})^{\frac{1}{2}}, b)\mathrm{d}r$$

$$\textit{where} \quad \Phi_n(x) = \begin{cases} \dfrac{1}{B\left(\frac{n-1}{2}, \frac{1}{2}\right)} \displaystyle\int_0^\pi e^{ix\cos\theta} \sin^{n-2}\theta \mathrm{d}\theta, & \textit{for } n > 1; \\ \cos x, & \textit{for } n = 1. \end{cases}$$

(2) $EX = \mu$, *for* $p \in \mathbb{R}$, *and* $a, b, \beta > 0$; $p, b, \beta > 0$, *when* $a = 0$; $a, \beta > 0$, $p < -\frac{1}{2\beta}$, *when* $b = 0$.

(3)

$$\operatorname{Var}\mathbf{X} = E(\mathbf{X} - E\mathbf{X})(\mathbf{X} - E\mathbf{X})^T = \frac{\Gamma\left(\frac{n+2}{2\beta}\right)\bar{K}_{p+\frac{1}{\beta}}(a, b)}{n\Gamma\left(\frac{n}{2\beta}\right)\bar{K}_p(a, b)}\Sigma$$

for $p \in \mathbb{R}$, *and* $a, b, \beta > 0$; $p, b, \beta > 0$, *when* $a = 0$; $a, \beta > 0$, $p < -\beta^{-1}$, *when* $b = 0$.

(4) *The multivariate asymmetric and excess kurtosis coefficients (see* [23], *p. 31 for more details), denoted as* γ_1 *and* γ_2 *respectively, are* $\gamma_1(\mathbf{X}) = E[((\mathbf{X} - E\mathbf{X})^T(\operatorname{Var}\mathbf{X})^{-1}(\mathbf{Y} - E\mathbf{Y}))^3] = 0$, *for* $p \in \mathbb{R}$, *and* $a, b, \beta > 0$; $p, b, \beta > 0$, *when* $a = 0$; $a, \beta > 0$, $p < -\frac{3}{2\beta}$, *when* $b = 0$; *and*

$$\begin{aligned}\gamma_2(\mathbf{X}) &= E[((\mathbf{X} - E\mathbf{X})^T(\operatorname{Var}\mathbf{X})^{-1}(\mathbf{X} - E\mathbf{X}))^2] - n(n+2)\\ &= \frac{n^2\Gamma\left(\frac{n+4}{2\beta}\right)\Gamma\left(\frac{n}{2\beta}\right)\bar{K}_p(a, b)\bar{K}_{p+\frac{2}{\beta}}(a, b)}{\left[\Gamma\left(\frac{n+2}{2\beta}\right)\bar{K}_{p+\frac{1}{\beta}}(a, b)\right]^2} - n(n+2),\end{aligned}$$

for $p \in \mathbb{R}$, *and* $a, b, \beta > 0$; $p, b, \beta > 0$, *when* $a = 0$; $a, \beta > 0$, $p < -\frac{2}{\beta}$, *when* $b = 0$; *where* $\mathbf{Y}$ *is a random vector independent and equally distributed as* $\mathbf{X}$. *The term* $n(n+2)$ *corresponds to the value of* $E[((\mathbf{X}-E\mathbf{X})^T(\operatorname{Var}\mathbf{X})^{-1}(\mathbf{X}-E\mathbf{X}))^2]$ *when* $\mathbf{X}$ *is distributed as a multivariate normal.*

Theorem 8. *If* $\mathbf{X}$ *is a random vector that* $\mathbf{X} \sim GGH_n(\mu_{\mathbf{X}}, \Sigma_{\mathbf{X}}, \beta, a, b, p)$ *and* $\mathbf{Y} = C\mathbf{X} + \gamma$.

(1) *If* C *is an* $n \times n$ *positive definite matrix and* $\gamma \in \mathbb{R}^n$ *then* $\mathbf{Y} \sim GGH_n(C\mu_{\mathbf{X}} + \gamma, C\Sigma_{\mathbf{X}}C^T, \beta, a, b, p)$.

(2) *If* C *is an* $m \times n$ *matrix with* $m < n$ *and rank*$(C) = m$ *and* $\gamma \in \mathbb{R}^m$ *then* $\mathbf{Y} \sim E_m(\mu_{\mathbf{Y}}, \Sigma_{\mathbf{Y}}, g_{\mathbf{Y}})$, *with* $\mu_{\mathbf{Y}} = C\mu_{\mathbf{X}} + \gamma$, $\Sigma_{\mathbf{Y}} = C\Sigma_{\mathbf{X}}C^T$ *and*

$$g_{\mathbf{Y}}(t^2) = \frac{\beta\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{m}{2}}\Gamma\left(\frac{n}{2\beta}\right)\Gamma\left(\frac{n-m}{2}\right)\bar{K}_p(a, b)}g_1(t^2), \tag{13}$$

where

$$g_1(t^2) = t^{n-m}\int_0^1 w^{\frac{m-n}{2}-1}(1-w)^{\frac{n-m}{2}-1}\bar{K}_{p-\frac{n}{2\beta}}\left(\left(a^2 + 2\left(\frac{t}{\sqrt{w}}\right)^{2\beta}\right)^{\frac{1}{2}}, b\right)\mathrm{d}w.$$

(3) *Partition* $\mathbf{X}$, μ *and* Σ *as* $\mathbf{X} = \binom{\mathbf{X}_1}{\mathbf{X}_2}$, $\mu = \binom{\mu_1}{\mu_2}$, $\Sigma = \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}$ *where* $\mathbf{X}_1$ *and* μ_1 *are* m *vectors and* Σ_{11} *is an* $m \times m$ *matrix, with* $m < n$. *Then* $\mathbf{X}_1 \sim E_m(\mu_1, \Sigma_{11}, g_{\mathbf{Y}})$, *where* $g_{\mathbf{Y}}(t^2)$ *is defined in* (13).

(4) *The conditional distribution of* $\mathbf{X}_2$ *given* $\mathbf{X}_1 = \mathbf{x}_1$ *is* $E_{n-m}(\mu_{2.1}, \Sigma_{22.1}, g_{2.1})$ *with* $\mu_{2.1} = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1)$, $\Sigma_{22.1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$ *and* $g_{2.1}(t^2) = k_{2.1}\bar{K}_{p-\frac{n}{2\beta}}((a^2 + 2((x_1 - \mu_1)^T\Sigma_{11}^{-1}(x_1 - \mu_1) + t^2)^\beta)^{\frac{1}{2}}, b)$, *where*

$$k_{2.1} = \frac{\Gamma(\frac{n-m}{2})}{\pi^{\frac{n-m}{2}}g_1((\mathbf{x}_1 - \mu_1)^T\Sigma_{11}^{-1}(\mathbf{x}_1 - \mu_1))}.$$

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References

[1] O. Arslan, Family of multivariate generalised t distribution, Journal of Multivariate Analysis 89 (2004) 329–337.
[2] O. Arslan, A new class of multivariate distribution: Scale mixture of Kotz-type distributions, Statistics and Probability Letters 75 (2005) 18–28.
[3] O. Arslan, Variance–mean mixture of Kotz-type distributions, Communications in Statistics-Theory and Methods 38 (2009) 272–284.
[4] S.W. Harrar, E. Seneta, A.K. Gupta, Duality between matrix variate t and matrix variate V.G. distributions, Journal of Multivariate Analysis 97 (2006) 1467–1475.
[5] D.B. Madan, E. Seneta, The variance-gamma (VG) model for share market returns, Journal of Business 63 (1990) 511–524.
[6] O. Barndorff-Nielsen, Exponential decreasing distributions for the logarithm of particle size, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences 353 (1990) 401–419.
[7] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Bateman Manuscript Project: Tables of Integral Transform, vol. 2, McGraw-Hill, New York, 1989.
[8] N.W. McLachlan, Bessel Functions for Engineers, 2nd ed., Clarendon Press, Oxford, 1955.
[9] S.R. Hurst, E. Platen, S.R. Rachev, Subordinated Markov models: A comparison, Financial Engineering and the Japanese Markets 4 (1997) 97–124.

[10] J.B. McDonald, W.K. Newey, Partially adaptive estimation of regression models via the generalised t distribution, Econometric Theory 4 (1988) 428–457.
[11] S.T.B. Choy, J.S.K. Chan, Sale mixtures distributions in statistical modelling, Australian & New Zealand Journal of Statistics 50 (2008) 135–146.
[12] O. Arslan, A.İ. Genç, Robust location and scale estimation based on the univariate generalised t (GT) distribution, Communications in Statistics-Theory and Methods 32 (2003) 1505–1525.
[13] E. Luciano, W. Schoutens, A multivariate jump-driven financial asset model, Quantitative Finance 6 (2006) 385–402.
[14] B. Jørgensen, Statistical properties of the generalized inverse Gaussian distribution for log-linear and stochastic volatility models, in: Lecture Notes in Statistics, vol. 9, Springer, New York, 1982.
[15] R. Silva, H. Lopes, H. Migon, The extended generalised inverse Gaussian distribution for log-linear and stochastic volatility models, Brazilian Journal of Probability and Statistics 20 (2006) 67–91.
[16] E.W. Stacy, A generalisation of the gamma distribution, Annals of Mathematical Statistics 33 (1962) 1187–1192.
[17] B.M. Steece, The distribution of the product and quotient for powers of two independent generalised gamma distribution, Metron 33 (1975) 227–235.
[18] K.T. Fang, S. Kotz, K.W. Ng, Symmetric Multivariate and Related Distributions, Chapman and Hall, London, 1990.
[19] E. Gómez, M.A. Gómez-Villegas, J.M. Marín, A multivariate generalisation of the power exponential family of distributions, Communications in Statistics-Theory and Methods 27 (1998) 589–600.
[20] K.T. Fang, Y.T. Zhang, Generalized Multivariate Analysis, Springer-Verlag, New York, 1990.
[21] S.J. Press, Multivariate stable distributions, Journal of Multivariate Analysis 2 (1972) 444–462.
[22] T. Fung, E. Seneta, A characterisation of scale mixture of the uniform distribution, Statistics and Probability Letters 78 (2008) 2883–2888.
[23] K.V. Mardia, J.T. Kent, J.M. Bibby, Multivariate Analysis, Academic Press, New York, 1979.